Microwave Measurement and Beam Instrumentation Course
at Jefferson Laboratory, January 15-26th 2018

U.S. Particle Accelerator School
Education in Beam Physics and Accelerator Technology

Lecture:
Maxwell's Equations

F. Marhauser

Monday, January 15, 2018

Jefferson Lab

U.S. DEPARTMENT OF ENERGY | Office of Science

JSA

This Lecture

- This lecture provides theoretical basics useful for follow-up lectures on resonators and waveguides

- Introduction to Maxwell’s Equations
  - Sources of electromagnetic fields
  - Differential form of Maxwell’s equation
  - Stokes’ and Gauss’ law to derive integral form of Maxwell’s equation
  - Some clarifications on all four equations
  - Time-varying fields → wave equation
  - Example: Plane wave
    - Phase and Group Velocity
    - Wave impedance

Maxwell's Equations

A dynamical theory of the electromagnetic field

James Clerk Maxwell, F. R. S.
Philosophical Transactions of the Royal Society of London, 1865 **155**, 459-512,
published 1 January 1865

PART I.—INTRODUCTORY.

(1) The most obvious mechanical phenomenon in electrical and magnetical experiments is the mutual action by which bodies in certain states set each other in motion while still at a sensible distance from each other. The first step, therefore, in reducing these phenomena into scientific form, is to ascertain the magnitude and direction of the force acting between the bodies, and when it is found that this force depends in a certain way upon the relative position of the bodies and on their electric or magnetic condition, it seems at first sight natural to explain the facts by assuming the existence of something either at rest or in motion in each body, constituting its electric or magnetic state, and capable of acting at a distance according to mathematical laws.

Jefferson Lab

Maxwell’s Equations

- Originally there were 20 equations

Three equations of magnetic force (H_x, H_y, H_z)

$$\mu H_x = \frac{dA_z}{dy} - \frac{dA_y}{dz}, \quad \mu H_y = \frac{dA_x}{dz} - \frac{dA_z}{dx} \quad \text{and} \quad \mu H_z = \frac{dA_y}{dx} - \frac{dA_x}{dy}.$$

Three equations of electric currents (J_x, J_y, J_z)

$$\frac{dH_z}{dy} - \frac{dH_y}{dz} = 4\pi J'_x, \quad \frac{dH_x}{dz} - \frac{dH_z}{dx} = 4\pi J'_y \quad \text{and} \quad \frac{dH_y}{dx} - \frac{dH_x}{dy} = 4\pi J'_z.$$

Three equations of electromotive force (E_x, E_y, E_z)

$$\left.\begin{aligned} E_x &= \mu\left(H_z\frac{dy}{dt} - H_y\frac{dz}{dt}\right) - \frac{dA_x}{dt} - \frac{d\phi}{dx}, \\ E_y &= \mu\left(H_x\frac{dz}{dt} - H_z\frac{dx}{dt}\right) - \frac{dA_y}{dt} - \frac{d\phi}{dy} \\ \text{and} \quad E_z &= \mu\left(H_y\frac{dx}{dt} - H_x\frac{dy}{dt}\right) - \frac{dA_z}{dt} - \frac{d\phi}{dz}. \end{aligned}\right\}$$

Three equations of electric elasticity (D_x, D_y, D_z)

$$E_x = kD_x, \quad E_y = kD_y \quad \text{and} \quad E_z = kD_z.$$

Three equations of electric resistance (ϱ)

$$E_x = -\varrho J_x, \quad E_y = -\varrho J_y \quad \text{and} \quad E_z = -\varrho J_z.$$

Three equations of total currents (J'_x, J'_y, J'_z)

$$J'_x = J_x + \frac{dD_x}{dt}, \quad J'_y = J_y + \frac{dD_y}{dt} \quad \text{and} \quad J'_z = J_z + \frac{dD_z}{dt}.$$

One equation of free electricity (ρ_e)

$$\rho_e + \frac{dD_x}{dx} + \frac{dD_y}{dy} + \frac{dD_z}{dz} = 0.$$

One equation of continuity $(d\rho_e/dt)$

$$\frac{d\rho_e}{dt} + \frac{dJ_x}{dx} + \frac{dJ_y}{dy} + \frac{dJ_z}{dz} = 0.$$

The result is 20 equations for the 20 variables which are:

electromagnetic momentum	A_x	A_y	A_z
magnetic intensity	H_x	H_y	H_z
electromotive force	E_x	E_y	E_z
current due to true conduction	J_x	J_y	J_z
electric displacement	D_x	D_y	D_z
total current (including variation of displacement)	J'_x	J'_y	J'_z
quantity of free electricity	ρ_e		
electric potential	ϕ		

contains Lorentz force

continuity equation

$$B = \mu H = \operatorname{curl} A, \tag{B}$$

$$\operatorname{curl} H = 4\pi J' = 4\pi\left(J + \frac{dD}{dt}\right), \tag{C}$$

$$E = \mu(v \times H) - \frac{dA}{dt} - \nabla\phi = (v \times B) - \frac{dA}{dt} - \nabla\phi, \tag{D}$$

$$E = kD, \tag{E}$$

$$E = -\varrho J, \tag{F}$$

$$J' = J + \frac{dD}{dt}, \tag{A}$$

$$\rho_e + \nabla \cdot D = 0 \tag{G}$$

$$\frac{d\rho_e}{dt} + \nabla \cdot J = 0. \tag{H}$$

Taken from Longair, M. 2015 ‘...a paper ...I hold to be great guns’: a commentary on Maxwell (1865) ‘A dynamical theory of the electromagnetic field’. Phil. Trans. R. Soc. A **373**: 20140473.

Sources of Electromagnetic Fields

- Electromagnetic fields arise from 2 sources:
  - Electrical charge (Q) → Stationary charge creates electric field
  - Electrical current ($I = \frac{dQ}{dt}$) → Moving charge creates magnetic field
- Typically charge and current densities are utilized in Maxwell's equations to quantify the effects of fields:
  - $\rho = \frac{dQ}{dV}$ electric charge density – total electric charge per unit volume V (or $Q = \iiint_V \rho \, dV$)
  - $J = \lim_{S \to 0} \frac{I(S)}{S}$ electric current density – total electric current per unit area S (or $I = \iint_S \vec{J} \cdot d\vec{S}$)
- If either the magnetic or electrical fields vary in time, both fields are coupled and the resulting fields follow Maxwell's equations

Maxwell's Equations

Differential Form

(1) $\vec{\nabla} \cdot \vec{D} = \rho$ or $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ Gauss's law

(2) $\vec{\nabla} \cdot \vec{B} = 0$ Gauss's law for magnetism

(3) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ Faraday's law of induction

(4) $\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$ or $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ Ampère's law

- Together with the Lorentz force these equations form the basic of the classic electromagnetism

$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$ Lorentz Force

$$\vec{D} = \epsilon_0 \vec{E}$$

ϵ_0=permittivity of free space

$$\vec{B} = \mu_0 \vec{H}$$

μ_0=permeability of free space

ρ = electric charge density (As/m^3)

J = electric current density (A/m^2)

D = electric flux density/displacement field (Unit: As/m^2)

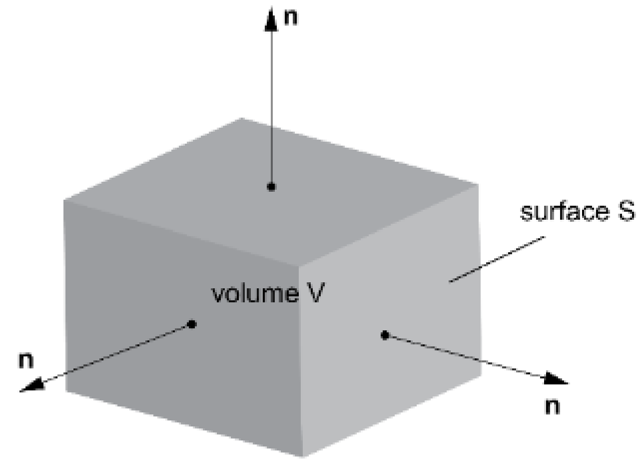
E = electric field intensity (Unit: V/m)

H = magnetic field intensity (Unit: A/m)

B = magnetic flux density (Unit: Tesla=Vs/m^2)

Divergence (Gauss') Theorem

$$\iiint_V \overbrace{(\nabla \cdot \vec{F})}^{\text{div}}\, dV = \oiint_S (\vec{F} \cdot \hat{n})\, dS = \oiint_S \vec{F} \cdot d\vec{S}$$

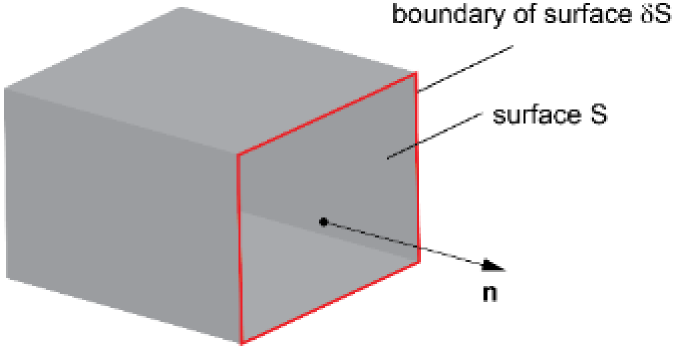


Integral of divergence of vector field ($\vec{F}$) over volume V inside closed boundary S **equals** outward flux of vector field ($\vec{F}$) through closed surface S

$$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(F_x, F_y, F_z\right) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Curl (Stokes') Theorem

$$\iint_S \underbrace{(\nabla \times \vec{F})}_{\text{curl}} \cdot d\vec{S} = \oiint_S ((\nabla \times \vec{F}) \cdot \hat{n}) dS = \oint_{\partial S} \vec{F} \cdot d\vec{l}$$



Integral of curl of vector field ($\vec{F}$) over surface S **equals**
line integral of vector field ($\vec{F}$) over closed boundary d*S* defined by surface S

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial F_y}{\partial z} - \frac{\partial F_x}{\partial y}\right)\hat{k}$$

$$\boldsymbol{e.g.: F_z = 0} \rightarrow \nabla \times \vec{F} = \left(\frac{\partial F_y(x,y)}{\partial x} - \frac{\partial F_x(x,y)}{\partial y}\right)\hat{k}$$

Curl vector is perpendicular to surface S ; $\hat{k} = \hat{n}$

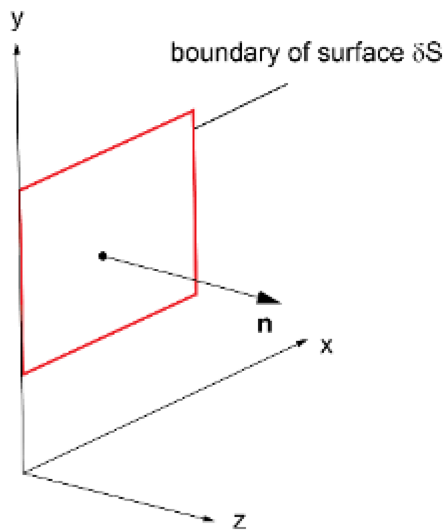
$$\iint_S \left(\frac{\partial F_y(x,y)}{\partial x} - \frac{\partial F_x(x,y)}{\partial y}\right) dS = \oint_{\partial S} F_x dx + F_y dy$$

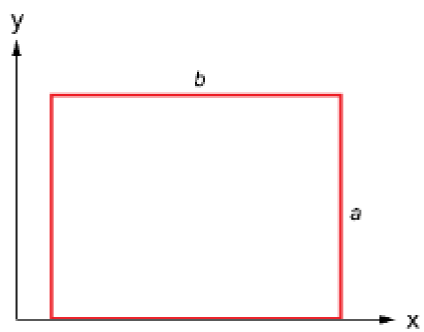
Green's Theorem

Example: Curl (Stokes') Theorem

$$\iint_S (\overbrace{\nabla \times \vec{F}}^{\text{curl}}) \cdot d\vec{S} = \oiint_S ((\nabla \times \vec{F}) \cdot \hat{n}) dS = \oint_{\partial S} \vec{F} \cdot d\vec{l}$$

Integral of curl of vector field ($\vec{F}$) over surface S **equals**
line integral of vector field ($\vec{F}$) over closed boundary d*S* defined by surface *S*



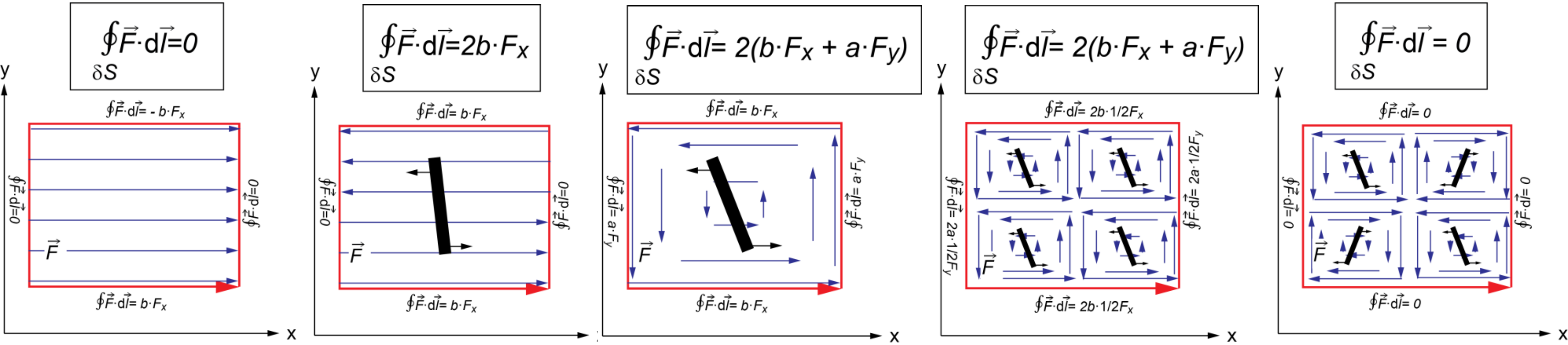


Example: Curl (Stokes) Theorem

$$\iint_S (\overbrace{\nabla \times \vec{F}}^{\text{curl}}) \cdot d\vec{S} = \oiint_S ((\nabla \times \vec{F}) \cdot \hat{n}) dS = \oint_{\partial S} \vec{F} \cdot d\vec{l}$$

Integral of curl of vector field ($\vec{F}$) over surface *S* **equals**
line integral of vector field ($\vec{F}$) over closed boundary d*S* defined by surface *S*

Example: Closed line integrals of various vector fields



No curl | Some curl | Stronger curl | No net curl

Jefferson Lab

Maxwell's Equations

Differential Form

$$\vec{\nabla} \cdot \vec{D} = \rho$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\iiint_V (\nabla \cdot \vec{F})\, dV = \oiint_S \vec{F} \cdot d\vec{S}$$

Gauss' theorem

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{l}$$

Stokes' theorem

Integral Form

$$\oiint_S \vec{D} \cdot d\vec{S} = \iiint_V \rho\, dV$$

Gauss's law

$$\oiint_S \vec{B} \cdot d\vec{S} = 0$$

Gauss's law for magnetism

$$\oint_{\partial S} \vec{E} \cdot d\vec{l} = -\iint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$

Faraday's law of induction

$$\oint_{\partial S} \vec{H} \cdot d\vec{l} = \iint_S \vec{J} \cdot d\vec{S} + \iint_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S}$$

Ampère's law

$$\vec{D} = \epsilon_0 \vec{E}$$

ϵ_0=permittivity of free space

$$\vec{B} = \mu_0 \vec{H}$$

μ_0=permeability of free space

ρ = electric charge density (C/m^3=As/m^3)

J = electric current density (A/m^2)

D = electric flux density/displacement field (Unit: As/m^2)

E = electric field intensity (Unit: V/m)

H = magnetic field intensity (Unit: A/m)

B = magnetic flux density (Unit: Tesla=Vs/m^2)

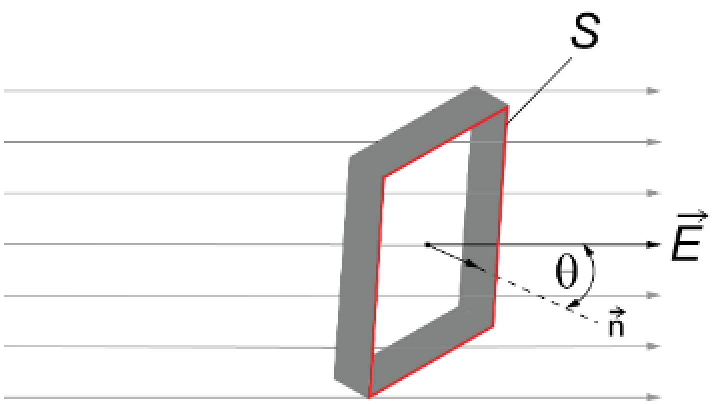
Electric Flux & 1st Maxwell Equation

Definition of Electric Flux

1. Uniform field

$$\Phi_E = \vec{E} \cdot \vec{S} = \vec{E} \cdot \hat{n}\, S = E \cdot S \cdot cos(\theta)[Vm]$$

- angle between field and normal vector to surface matters



2. Non-Uniform field

$$d\Phi_E = \vec{E} \cdot d\vec{S} = \vec{E} \cdot \hat{n}\, dS$$

$$\Phi_E = \iint_S \vec{E} \cdot d\vec{S}$$

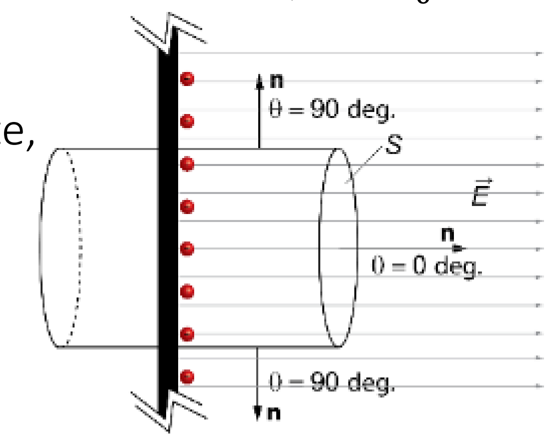
Gauss: Integration over **closed** surface

$$\boxed{\oiint_S \epsilon_0 \vec{E} \cdot d\vec{S} = \epsilon_0\, \Phi_{\boldsymbol{E}} = \iiint_V \rho\, dV = \sum_i q_i}$$

$$\boxed{\Phi_E = \frac{\sum_i q_i}{\epsilon_0}}$$

$; \vec{D} = \epsilon_0 \vec{E}$

Example: Metallic plate, assume only surface charges on one side



infinitely long metal plate

$$\boxed{\oiint_S \epsilon_0 \vec{E} \cdot d\vec{S} = \epsilon_0 |E| \pi R^2 = \sum_i q_i = Q_{circle}}$$

$$\boxed{|E| = \frac{Q_{circle}}{\epsilon_0 \pi R^2}}$$

$; circle\ S = \pi R^2$

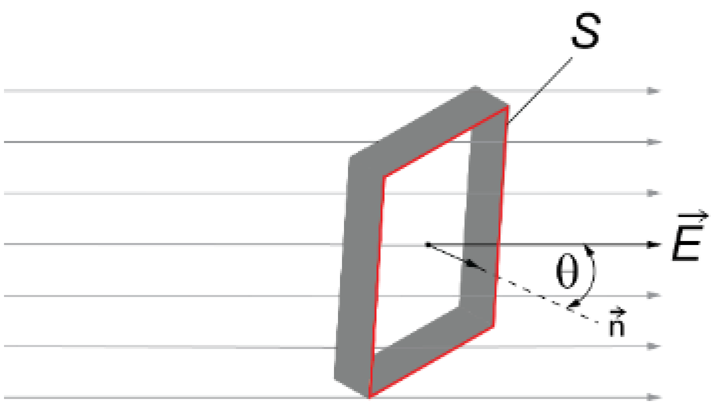
Electric Flux & 1st Maxwell Equation

Definition of Electric Flux

1. Uniform field

$$\Phi_E = \vec{E} \cdot \vec{S} = \vec{E} \cdot \hat{n}\, S = E \cdot S \cdot cos(\theta)[Vm]$$

- angle between field and normal vector to surface matters



2. Non-Uniform field

$$d\Phi_E = \vec{E} \cdot d\vec{S} = \vec{E} \cdot \hat{n}\, dS$$

$$\Phi_E = \iint_S \vec{E} \cdot d\vec{S}$$

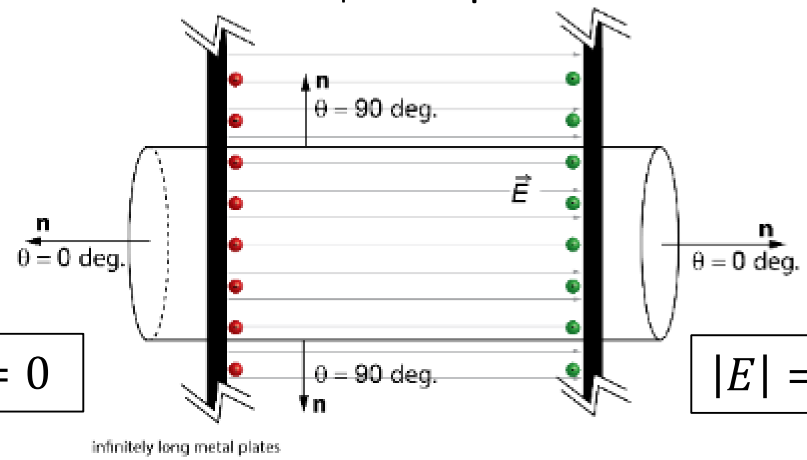
Gauss: Integration over **closed** surface

$$\boxed{\oiint_S \epsilon_0 \vec{E} \cdot d\vec{S} = \epsilon_0\, \Phi_{\boldsymbol{E}} = \iiint_V \rho\, dV = \sum_i q_i}$$

$$\boxed{\Phi_E = \frac{\sum_i q_i}{\epsilon_0}}$$

$; \vec{D} = \epsilon_0 \vec{E}$

Example: **Capacitor**



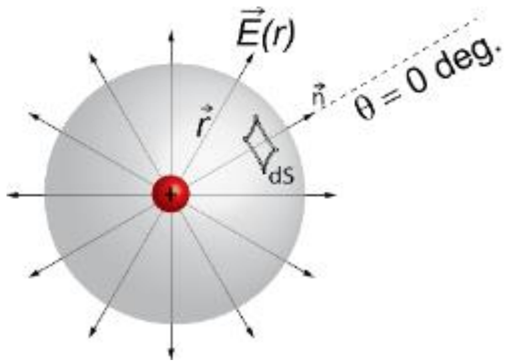
$\boxed{|E| = 0}$ $\boxed{|E| = 0}$

$$\boxed{\Phi_E = \frac{Q_{circle}}{\epsilon_0} + \frac{-Q_{circle}}{\epsilon_0} = 0}$$

Electric Flux & 1st Maxwell Equation

Examples of non-uniform fields

Point charge Q



Integration of over closed spherical surface S

$$\oiint_S \epsilon_0 \vec{E} \cdot d\vec{S} = \epsilon_0 \boldsymbol{E}(r) 4\pi r^2 = Q$$

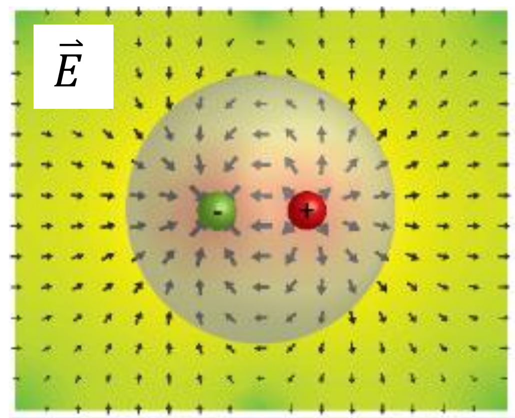
$; sphere\ S = 4\pi r^2$

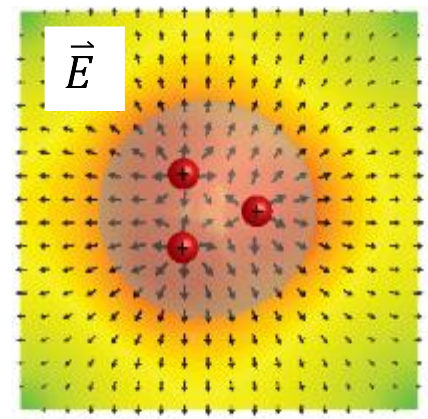
pointing out radially

$$\boldsymbol{E}(r) = \frac{Q}{\epsilon_0 4\pi r^2} \cdot \hat{\boldsymbol{r}}$$

Add charges

$$\oiint_S \epsilon_0 \vec{E} \cdot d\vec{S} = \iiint_V \rho\, dV = \sum_i q_i = Q_{sphere}$$





$$\Phi_E = \frac{\sum_i q_i}{\epsilon_0} = \frac{q}{\epsilon_0} + \frac{-q}{\epsilon_0} = 0$$

$$\Phi_E = \frac{3q}{\epsilon_0} = \frac{Q_{sphere}}{\epsilon_0}$$

Principle of Superposition holds:

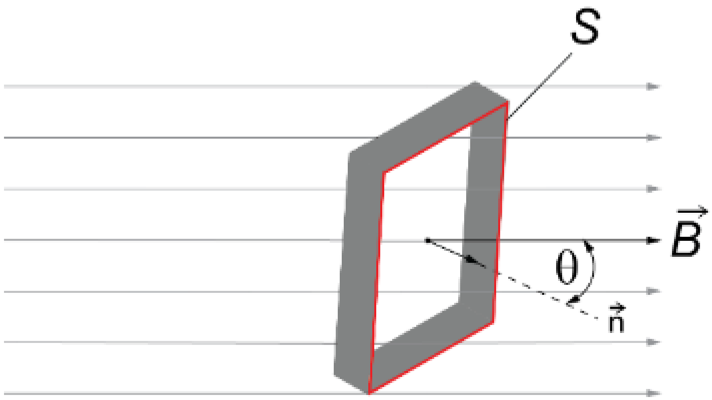
$$\vec{E}(r) = \frac{1}{\epsilon_0 4\pi}\left(\frac{q_1}{(r_{c1}-r)^2}\hat{r}_{c1} + \frac{q_2}{(r_{c2}-r)^2}\hat{r}_{c2} + \frac{q_3}{(r_{c3}-r)^2}\hat{r}_{c3} + \cdots\right)$$

Magnetic Flux & 2nd Maxwell Equation

Definition of Magnetic Flux

Uniform field

$$\Phi_B = \vec{B} \cdot \vec{S} = \vec{B} \cdot \hat{n}\, S = B \cdot S\, cos(\theta)[Wb = Vs]$$



Non-Uniform field

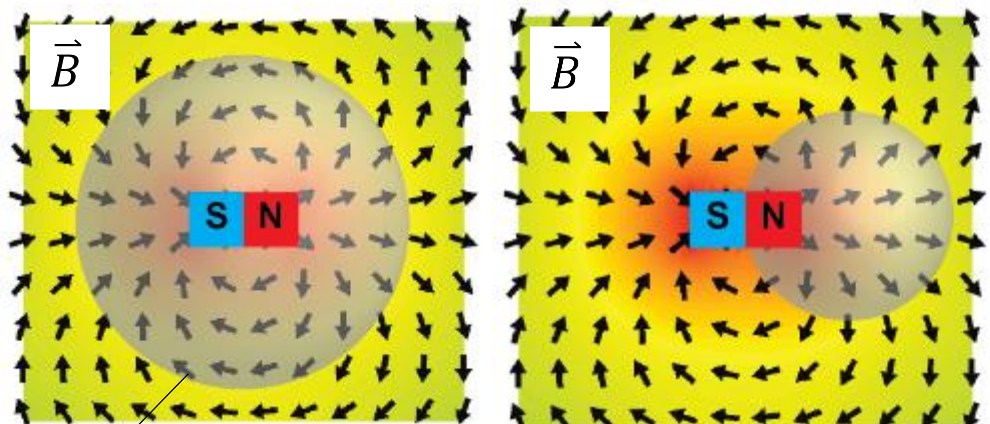
$$d\Phi_B = \vec{B} \cdot d\vec{S} = \vec{B} \cdot \hat{n}\, dS$$

$$\Phi_B = \iint_S \vec{B} \cdot d\vec{S}$$

Gauss: Integration over **closed** surface

$$\boxed{\oiint_S \vec{B} \cdot d\vec{S} = 0} \qquad \boxed{\Phi_M = 0}$$

- There are **no magnetic monopoles**
- All magnetic field lines form loops



Closed surface:
Flux lines out = flux lines in

What about this case?
Flux lines out > flux lines in ?

- No. In violation of 2nd Maxwell's law, i.e. integration over closed surface, no holes allowed
- Also: One cannot split magnets into separate poles, i.e. there always will be a North and South pole

Magnetic Flux & 3rd Maxwell Equation

If integration path is not changing in time

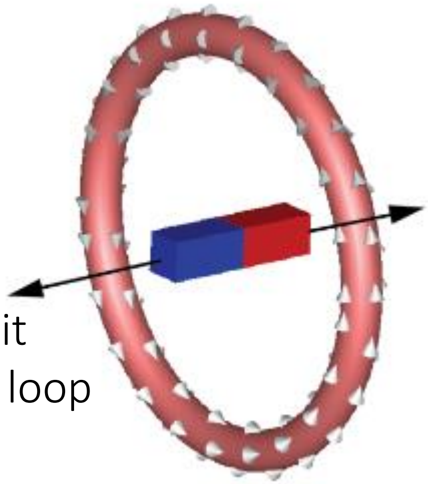
$$\oint_{\partial S} \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \iint_{S} \vec{B} \cdot d\vec{S} = -\frac{d\Phi_B}{dt}$$

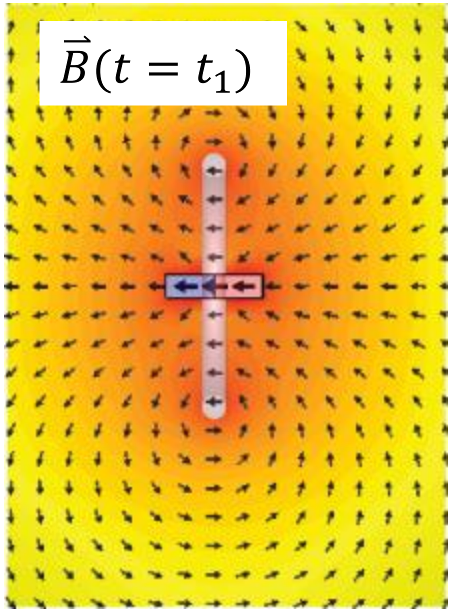
$; \Phi_B = \iint_S \vec{B} \cdot d\vec{S}$

- Change of magnetic flux induces an electric field along a closed loop
- Note: Integral of electrical field over closed loop may be non-zero, when induced by a time-varying magnetic field
- Electromotive force (EMF) Ɛ:

$$\mathcal{E} = \oint_{\partial S} \vec{E} \cdot d\vec{l} \; [V]$$

- Ɛ equivalent to energy per unit charge traveling once around loop





Faraday’s law of induction

Magnetic Flux & 3rd Maxwell Equation

If integration path is not changing in time

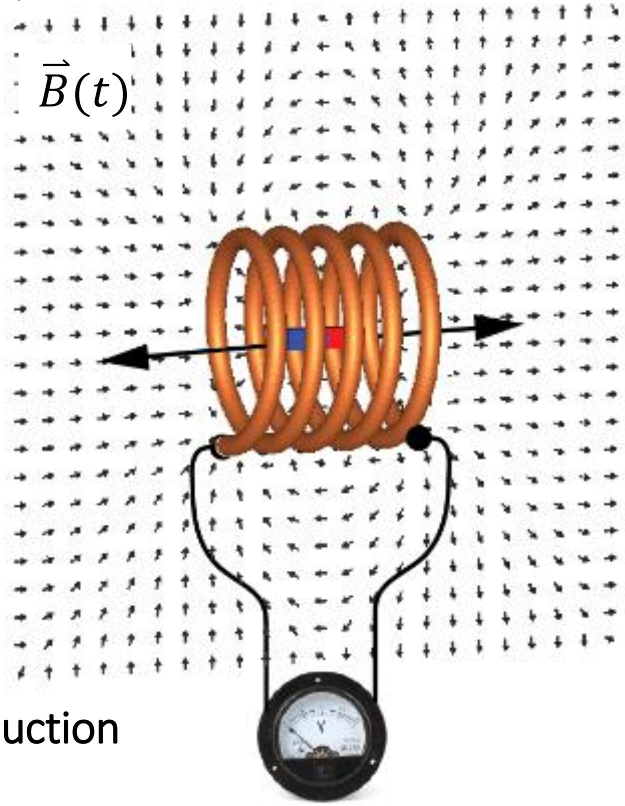
$$\oint_{\partial S} \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \iint_{S} \vec{B} \cdot d\vec{S} = -\frac{d\Phi_B}{dt}$$

$; \Phi_B = \iint_S \vec{B} \cdot d\vec{S}$

- Change of magnetic flux induces an electric field along a closed loop
- Note: Integral of electrical field over closed loop may be non-zero, when induced by a time-varying magnetic field
- Electromotive force (EMF) Ɛ:

$$\varepsilon = \oint_{\partial S} \vec{E} \cdot d\vec{l} \; [V]$$

- Ɛ equivalent to energy per unit charge traveling once around loop
- or voltage measured at end of open loop



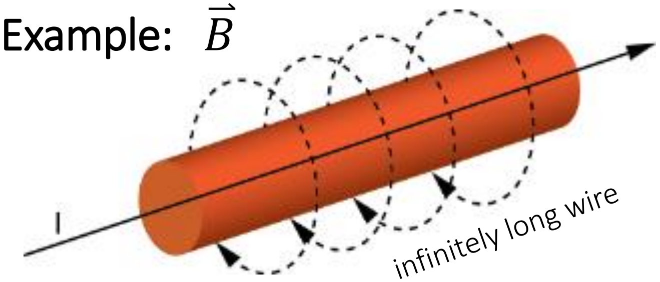
Faraday's law of induction

Ampère's (circuital) Law or 4th Maxwell Equation

$$\boxed{\oint_{\partial S} \vec{H} \cdot d\vec{l} = \underbrace{\iint_S \vec{J} \cdot d\vec{S}}_{\text{conduction current I}} + \iint_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S}}$$

conduction current I

Example: $\vec{B}$



Left hand side of equation:

$$\boxed{\oint_{\partial S} \vec{B} \cdot d\vec{l} = \boldsymbol{B}(r)\, 2\pi r = \mu_0 I}$$

$; \vec{B} = \mu_0 \vec{H}$

tangential to a circle at any radius r of integration

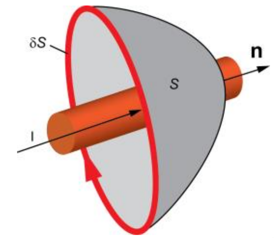
$; circumference\ C = 2\pi r$

$$\boxed{|\boldsymbol{B}(r)| = \frac{\mu_0 I}{2\pi r}}$$

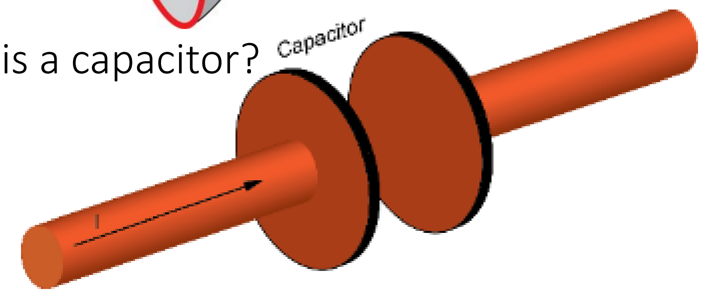
Right hand side of equation:

- Note that $\iint_S \vec{J} \cdot d\vec{S}$ is a surface integral, but *S* may have arbitrary shape as long as ∂S is its closed boundary

$$\iint_S \vec{J} \cdot d\vec{S} = I$$

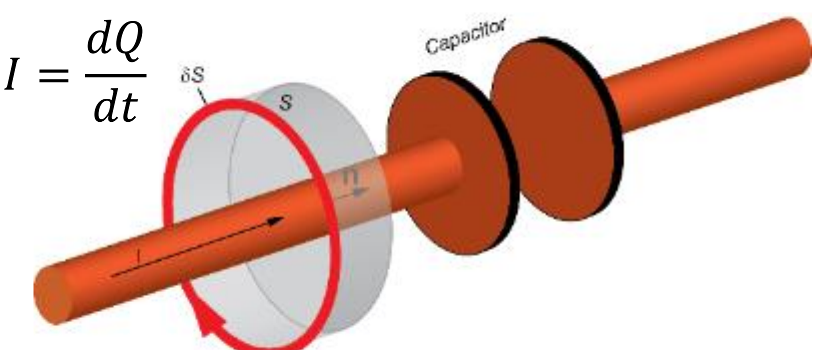


- What if there is a capacitor?



- While current is still be flowing (charging capacitor):

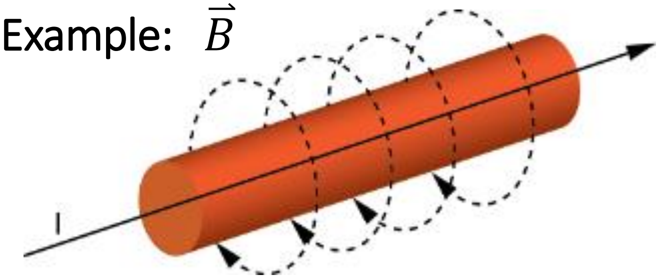
$$\iint_S \vec{J} \cdot d\vec{S} = I = \frac{dQ}{dt}$$



Ampère's (circuital) Law or 4th Maxwell Equation

$$\oint_{\partial S} \vec{H} \cdot d\vec{l} = \underbrace{\iint_S \vec{J} \cdot d\vec{S}}_{\text{conduction current I}} + \iint_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S}$$

Example: $\vec{B}$



Left hand side of equation:

$$\oint_{\partial S} \vec{B} \cdot d\vec{l} = \boldsymbol{B}(r)\, 2\pi r = \mu_0 I$$

tangential to a circle at any radius r of integration

$; \vec{B} = \mu_0 \vec{H}$

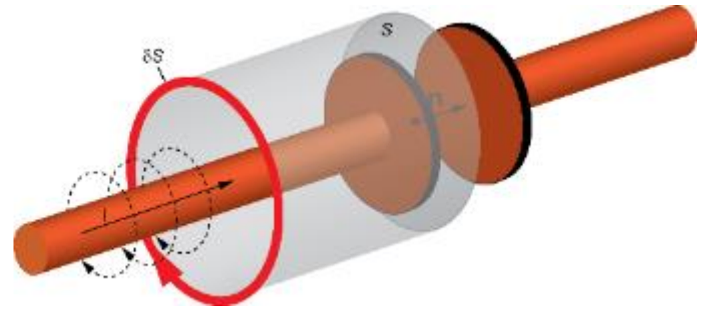
$; circumference\ C = 2\pi r$

$$\boxed{|\boldsymbol{B}(r)| = \frac{\mu_0 I}{2\pi r}}$$

- But one may also place integration surface S between plates → current does not flow through surface here

$$\iint_S \vec{J} \cdot d\vec{S} = 0$$

$while\ \vec{B} \neq 0\ ?$

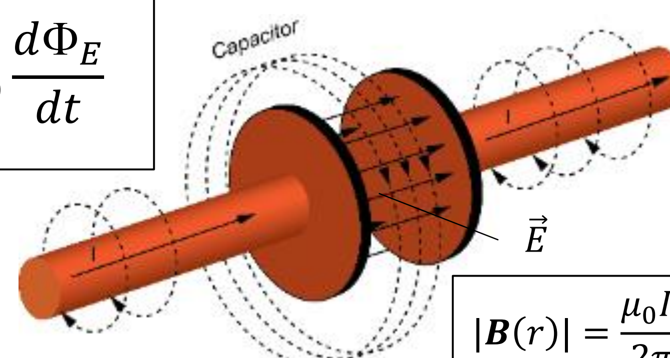


- This is when the displacement field is required as a corrective 2nd source term for the magnetic fields

$$\oiint_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S} = \frac{d}{dt} \oiint_S \vec{D} \cdot d\vec{S} = \frac{dQ}{dt} = I_D = \underbrace{\epsilon_0 \frac{d\Phi_E}{dt}}_{\text{displacement current I}}$$

; Gauss's law

$$\boxed{\oint_{\partial S} \vec{H} \cdot d\vec{l} = I + \epsilon_0 \frac{d\Phi_E}{dt}}$$



$$\boxed{|\boldsymbol{B}(r)| = \frac{\mu_0 I_D}{2\pi r}}$$

Presence of Resistive Material

$$\oint_{\partial S} \vec{H} \cdot d\vec{l} = \underbrace{\iint_S \vec{J} \cdot d\vec{S}}_{\text{conduction current}} + \underbrace{\iint_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S}}_{\text{displacement current}}$$

- In resistive materials the current density *J* is proportional to the electric field

$$\boxed{\vec{J} = \sigma \vec{E} = \frac{1}{\rho} \vec{E}}$$

with σ the electric conductivity (1/(Ω·m) or S/m), respectively
$\rho = 1/\sigma$ the electric resistivity (Ω·m)

- Generally $\sigma(\omega, T)$ is a function of frequency and temperature

Time-Varying E-Field in Free Space

- Assume charge-free, homogeneous, linear, and isotropic medium
- We can derive a wave equation:

$$\boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$;Faraday's law of induction

$$\boxed{\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\vec{\nabla} \times \frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})}$$; || curl

$\nabla^2 = \Delta$ = Laplace operator

$$\boxed{\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t}\left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)}$$; curl of curl $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$

; $\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$; Ampère's law

$$\boxed{\vec{\nabla}\left(\frac{\rho}{\epsilon_0}\right) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t}\left(\sigma \vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t}\right)}$$

; Gauss's law ; $\vec{J} = \sigma \vec{E}$

$$\boxed{\nabla^2 \vec{E} - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0}$$ Homogeneous wave equation

; we presumed no charge

Jefferson Lab

Time-Varying B-Field in Free Space

- Assume charge-free, homogeneous, linear, and isotropic medium
- We can derive a wave equation:

$$\boxed{\vec{\nabla} \times \vec{B} = \mu \vec{J} + \mu\epsilon \frac{\partial \vec{E}}{\partial t}}$$; Ampère's law

$$\boxed{\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \mu(\vec{\nabla} \times \vec{J}) + \mu\epsilon \left(\vec{\nabla} \times \frac{\partial \vec{E}}{\partial t}\right)}$$; | | curl

$$\boxed{\vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \nabla^2 \vec{B} = \mu(\vec{\nabla} \times \vec{J}) - \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2}}$$; curl of curl $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$

; Faraday's law

$$\boxed{\nabla^2 \vec{B} - \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2} = 0}$$ Similar homogeneous wave equation as for *E*-Field

; Gauss's law for magnetism $\vec{\nabla} \cdot \vec{B} = 0$
; no moving charge ($\vec{J}=0$)

Time-Harmonic Fields

- In many cases one has to deal with purely harmonic fields ($\sim e^{i\omega t}$)

$$\boxed{\nabla^2 \vec{B} - \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2} = 0} \longrightarrow \boxed{\nabla^2 \vec{B} = -\mu\epsilon\omega^2 \vec{B}}$$

$$\boxed{\nabla^2 \vec{E} - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0} \longrightarrow \boxed{\nabla^2 \vec{E} = -\mu\epsilon\omega^2 \vec{E}}$$

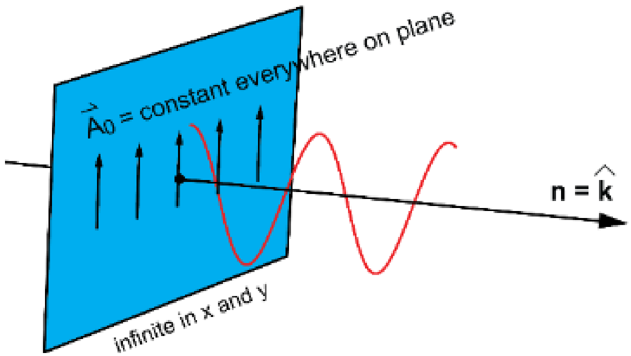
Jefferson Lab

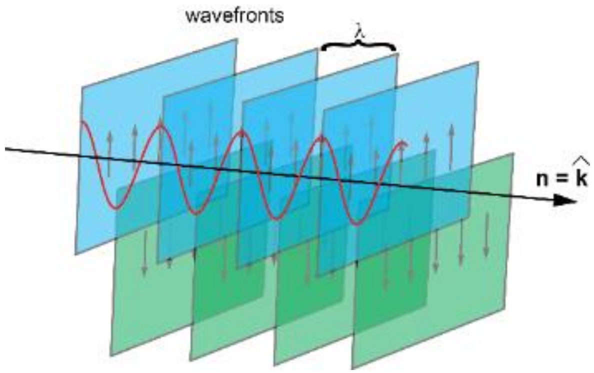
Example: Plane Wave in Free Space

$$\boxed{\vec{A}(\vec{r},t) = \vec{A}_0 \cdot e^{-i\,\vec{k}\cdot\vec{r}} \cdot e^{i\omega t} = \vec{A}_0 \cdot e^{i(\omega t - \vec{k}\cdot\vec{r})}}$$

$$|\vec{A}(\vec{r},t)| = Re(\vec{A})$$

- ***k*** is a wave vector pointing in direction of wave propagation
- Wave is unconstrained in plane orthogonal to wave direction, i.e. has surfaces of constant phase (wavefronts), wave vector ***k*** is perpendicular to the wavefront





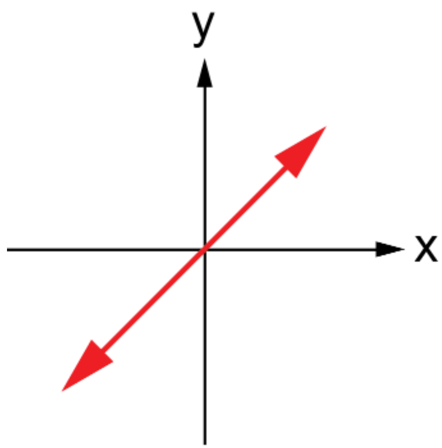
- Magnitude of field (whether it is ***E*** or ***B***) is constant everywhere on plane, but varies with time and in direction of propagation
- One may align propagation of wave (***k***) with z-direction, which simplifies the equation
- In Cartesian coordinates: $\boxed{\vec{A}(x,y,z,t) = \vec{A}_0 \cdot e^{-ikz} \cdot e^{i\omega t}}$
- Applying homogeneous wave equation $\nabla^2\vec{A} = -\mu\epsilon\omega^2\vec{A}$ (with $\nabla^2\vec{A} = \frac{\partial A_x}{\partial x^2} + \frac{\partial A_y}{\partial y^2} + \frac{\partial A_z}{\partial z^2}$

$$\boxed{\nabla^2\vec{A} = -k^2\vec{A} = -\mu\epsilon\omega^2\vec{A}} \rightarrow \boxed{k^2 = \mu\epsilon\omega^2} \rightarrow \boxed{k^2 = \mu\epsilon\omega^2 = \frac{\omega^2}{v^2}}$$

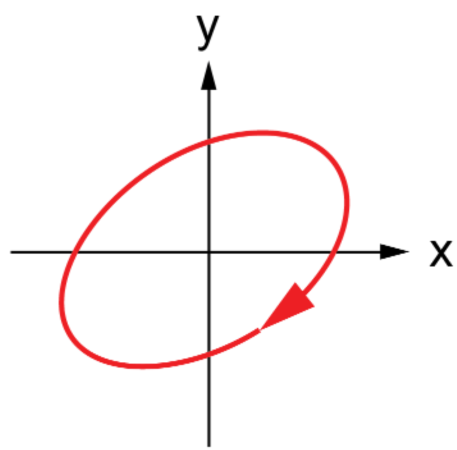
- We know **speed of light** in linear medium: $\boxed{v = \frac{1}{\sqrt{\mu\epsilon}}}$

Jefferson Lab

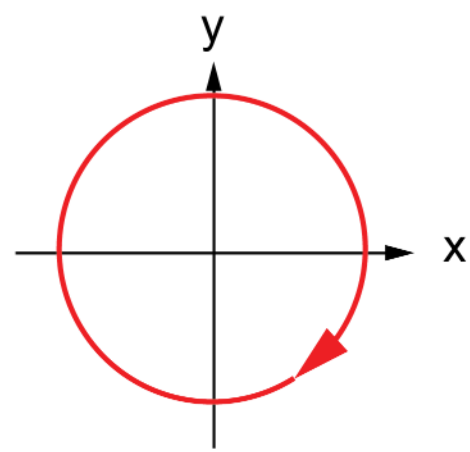
Example: Plane Wave in Free Space



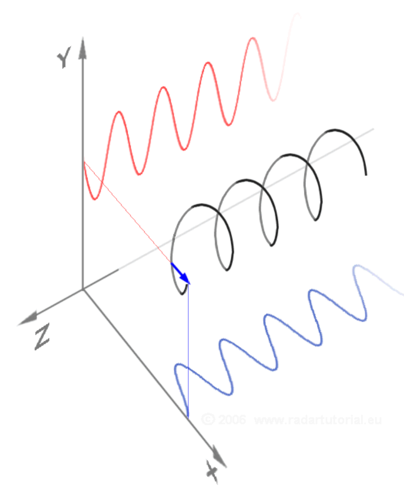
linear polarization
(can be superposition of horizontally
and vertically polarized wave
with same amplitude and phase)



elliptical polarization
(superposition of two lineared
polarizations with phase shift
between waves)



circular polarization
(similar to elliptical polarization
but with phase shift of +- 90 deg.
between waves)



Example: Plane Wave in Free Space

- Acknowledging that ***k*** is generally a vector: $\boxed{\vec{k} = k \cdot \widehat{k_z}}$
- Inserting the just derived equation $k^2 = \frac{\omega^2}{v^2}$, i.e. a dependency with the angular frequency, we can denote the relation of ***k*** with the wavelength

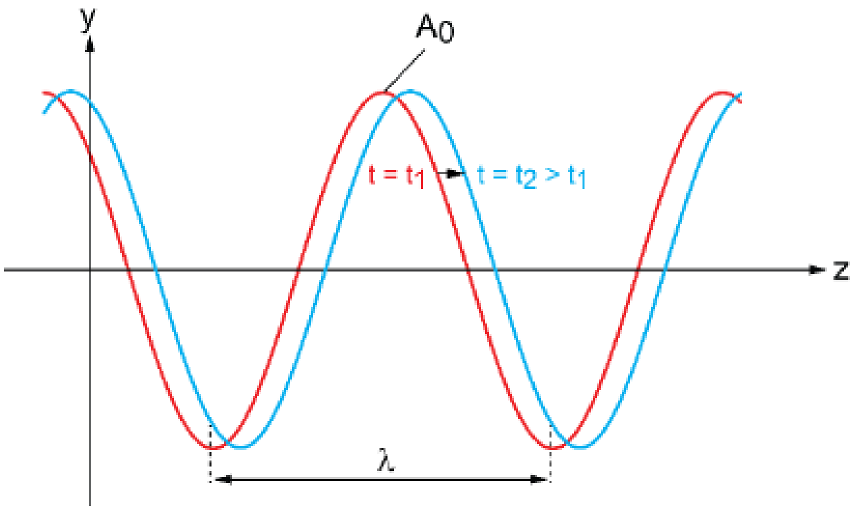
$$\boxed{\vec{k} = \frac{2\pi f}{v}\widehat{k_z} = \frac{2\pi}{\lambda}\widehat{k_z}}$$

$$\boxed{k = \frac{2\pi}{\lambda} = \frac{2\pi f}{v}}$$

k is the wavenumber [1/m]

$$\boxed{v_{ph} \equiv \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}} = c_0\frac{1}{\sqrt{\mu_r\epsilon_r}}}$$

Phase velocity



$$\boxed{v_{gr} \equiv \frac{d\omega}{dk} = \frac{1}{\sqrt{\mu\epsilon}} = v_{ph}}$$

Group velocity = Phase velocity = speed of light

Wave Impedance

- Furthermore for plane wave, due to 3rd Maxwell equation we know that magnetic field is orthogonal to electrical field and can derive for time-harmonic field:

$$\boxed{\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} = -\mu\omega\vec{H}}$$

- Considering the absence of charges in free space and 4th Maxwell equation, we find:

$$\boxed{\vec{\nabla} \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} = \varepsilon\omega\vec{E}}$$

- We then can find for the electrical field components considering

$$;\nabla \times \vec{F} = \left(\cancel{\frac{\partial F_z}{\partial y}} - \frac{\partial F_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial F_x}{\partial z} - \cancel{\frac{\partial F_z}{\partial x}}\right)\hat{\jmath} + \cancel{\left(\frac{\partial F_y}{\partial z} - \frac{\partial F_x}{\partial y}\right)}\hat{k}$$

$$\boxed{-\frac{\partial E_y}{\partial z}\hat{\imath} + \frac{\partial E_x}{\partial z}\hat{\jmath} = -i\mu\omega\vec{H} = -i\mu\omega H_x\hat{\imath} - i\mu\omega H_y\hat{\jmath}}$$

$$\frac{\partial E_y}{\partial z} = i\mu\omega H_x$$

$$\frac{\partial E_x}{\partial z} = -i\mu\omega H_y$$

- Similarly for the magnetic field considering

$$\boxed{-\frac{\partial H_y}{\partial z}\hat{\imath} + \frac{\partial H_x}{\partial z}\hat{\jmath} = i\varepsilon\omega\vec{E} = i\varepsilon\omega E_x\hat{\imath} + i\varepsilon\omega E_y\hat{\jmath}}$$

$$\frac{\partial H_y}{\partial z} = -i\varepsilon\omega E_x$$

$$\frac{\partial H_x}{\partial z} = i\varepsilon\omega E_y$$

- All field components are orthogonal to propagation direction
  → this means that the plane wave is a Transverse-Electric-Magnetic (**TEM**) wave

Wave Impedance

- We obtained two sets of independent equations, that lead to two linearly independent solutions

1a) $\frac{\partial E_x}{\partial z} = -i\mu\omega H_y$

1b) $\frac{\partial H_y}{\partial z} = -i\varepsilon\omega E_x$

2a) $\frac{\partial E_y}{\partial z} = i\mu\omega H_x$

2b) $\frac{\partial H_x}{\partial z} = i\varepsilon\omega E_y$

- The wave equation for the electric field components yields:

$$\frac{\partial^2 E_x}{\partial x^2} = -k^2 E_x \qquad \frac{\partial^2 E_y}{\partial x^2} = -k^2 E_y$$

$;\nabla^2 \vec{A} = \frac{\partial A_x}{\partial x^2} + \frac{\partial A_y}{\partial y^2} + \frac{\partial A_z}{\partial z^2}$

- Utilizing the Ansatz: $E_x = E_{x,p} e^{-ikz} + E_{x,r} e^{+ikz}$ $\quad E_y = E_{y,p} e^{-ikz} + E_{y,r} e^{+ikz}$

we can derive the corresponding magnetic field components:

$$H_y = \frac{k}{\mu\omega}\left(E_{x,p} e^{-ikz} - E_{x,r} e^{+ikz}\right) \quad ;1a)$$

$$H_x = -\frac{k}{\mu\omega}\left(E_{y,p} e^{-ikz} - E_{y,r} e^{+ikz}\right) \quad ;2a)$$

- Using the substitution $Z = \frac{k}{\mu\omega}$:

$$H_y = \frac{1}{Z}\left(E_{x,p} e^{-ikz} - E_{x,r} e^{+ikz}\right)$$

$$H_x = -\frac{1}{Z}\left(E_{y,p} e^{-ikz} - E_{y,r} e^{+ikz}\right)$$

$$Z = \frac{\mu\omega}{k} = \frac{\mu\omega}{\sqrt{\mu\epsilon}\,\omega} = \sqrt{\frac{\mu}{\epsilon}} \approx \sqrt{\frac{\mu_0}{\varepsilon_0}}\sqrt{\frac{\mu_r}{\varepsilon_r}} \quad ;\ k^2 = \mu\epsilon\omega^2$$

$$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 120\pi\ \Omega \approx 376.73\ \Omega$$

Z is the **wave impedance** in Ohms

vacuum impedance

Appendix

Presence of Dielectric Material

- For **linear** materials

$$\boxed{\epsilon = \epsilon_r \epsilon_0}$$

ε_r is relative permittivity

$$\boxed{\mu = \epsilon_r \epsilon_0}$$

μ_r is relative permeability

- Particularly, the displacement current was conceived by Maxwell as the separation (movement) of the (bound) charges due to the polarization of the medium (bound charges slightly separate inducing electric dipole moment)

$$\boxed{\vec{D} = \epsilon \vec{E} = \epsilon_0 \vec{E} + \vec{P}}$$

P is polarization density ('polarization') is the density of permanent and induced electric dipole moments

- For homogeneous, linear isotropic dielectric material

$$\boxed{\vec{P} = \epsilon_0 (\epsilon_r - 1) \vec{E}} \qquad \boxed{(\epsilon_r - 1) = \mathrm{c}_e}$$

c_e= electric susceptibility

- For anisotropic dielectric material

$$\boxed{\vec{P} = \sum_j \epsilon_0 \, \mathrm{c}_{i,j} \vec{E}_j}$$

- Material may be non-linear, i.e. ***P*** is not proportional to ***E***(→ hysteresis in ferroelectric materials)

- Generally ***P***(ω) is a function of frequency, since the bound charges cannot act immediately to the applied field (c_e(ω) → this gives rise to losses

Similar Expressions for Magnetization

- For magnetic fields the presence of magnetic material can give rise to a magnetization by microscopic electric currents or the spin of electrons
- Example: If a ferromagnet (e.g. iron) is exposed to a magnetic field, the microscopic dipoles align with the field and remain aligned to some extent when the magnetic field vanishes (magnetization vector *M*) → a non-linear dependency between *H* and *M* occurs
- Magnetization may occur in directions other than that of the applied magnetic field
- The magnetization vector describes the density of the permanent or induced magnetic dipole moments in a magnetic material

$$\boxed{\vec{B} = \mu_0 \cdot \vec{H} = \mu_0\left(\vec{H} + \vec{M}\right) = \mu_0(1 + \mathcal{X}_v)\vec{H}}$$

- Herein $\mathcal{X}_v$ is the magnetic susceptibility, which described whether is material if appealed or retracted by the presence of a magnetic field
- The relative permeability of the material can then be denoted as:

$$\boxed{\mu_r = 1 + \mathcal{X}_v}$$